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Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra

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Abstract

In the present paper we consider the Volterra integration operator V on the Wiener algebra $W(\mathbb{D})$ of analytic functions on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. A complex number λ is called an extended eigenvalue of V if there exists a nonzero operator A satisfying the equation $AV = \lambda VA$. We prove that the set of all extended eigenvalues of V is precisely the set $\mathbb{C}\backslash\{0\}$, and describe in terms of Duhamel operators and composition operators the set of corresponding extended eigenvectors of V. The similar result for some weighted shift operator on ℓ_p spaces is also obtained.

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1. Introduction and background

Denote by $\mathscr{B}(X)$ the algebra of all bounded linear operators on a Banach space X. Let $C \in \mathscr{B}(X)$ be a fixed operator. It can be happen that there are nonzero operators $A, B \in \mathscr{B}(X)$ such that

$$AC = CB. \tag{1}$$

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If we denote by $\mathscr{E}_C$ the set of all A for which there exists an operator B satisfying (1), then it is easy to see that $\mathscr{E}_C$ is an algebra. Furthermore, one can define the map $\Phi_C : \mathscr{E}_C \to \mathscr{B}(X)$ by $\Phi_C(A) = B$. One can easily see that Φ_C is an algebra homomorphism, and it can be verified that (see [1]) it is in fact a closed (generally unbounded) linear transformation.

When $B = \lambda A$, for some complex number λ, Eq. (1) becomes

$$AC = \lambda CA. \tag{2}$$

Clearly, a pair (A, λ) in $\mathscr{B}(X)\backslash\{0\} \times \mathbb{C}$ satisfies (2) if and only if λ is an eigenvalue for Φ_C and A is an eigenvector for Φ_C. Following[1], an eigenvalue of Φ_C will be referred to as an extended eigenvalue of C.

One knows that, when $\lambda = 1$, Eq. (2) can be used to obtain information about the operator A based on the properties of the operator C. In particular, a famous result of Lomonosov [8] asserts that if C is compact then A must have a nontrivial hyperinvariant subspace, that is the whole commutant $\{C\}'$ of C has a common nontrivial invariant subspace. Later, it was shown independently by Brown [2] and Kim et al. [9] that if C is compact and A satisfies (2), for any number $\lambda \in \mathbb{C}$, then A has a nontrivial hyperinvariant subspace. This extension naturally leads to the question as to whether there is an algebra $\mathscr{A}$ that properly contains $\{A\}'$ and which, under specific conditions, has an invariant subspace. Such an algebra has been introduced by Lambert and Petrovic [7] and it was shown that it contains not only those operators that commute with C but also operators that satisfy (2) for some $|\lambda| \leqslant 1$. (For the related results see also [10,5].) Furthermore, if C is a compact operator, then this algebra has a nontrivial invariant subspace. Certainly, if $\mathscr{A} = \{C\}'$ this is just Lomonosov's theorem. Therefore, it is of interest to find out whether the inclusion

$$\{C\}' \subset \mathscr{A} \tag{3}$$

is proper. It was established by Lambert and Petrovic [7] that this happens when the spectral radius of A is positive. Thus, it remains to consider the case in which C is compact and quasinilpotent.

A first step in this direction was made by Biswas et al. [1] by showing that inclusion (3) is proper when C is a specific compact, quasinilpotent operator (i.e., Volterra operator). More precisely, for $X = L^2(0, 1)$ and $C = V$, where V is the Volterra integration operator on $L^2(0, 1)$, defined by

$$(Vf)(x) = \int_0^x f(t)\,\mathrm{d}t.$$

It was shown in [1] that the set of all extended eigenvalues of V is precisely the set $(0, \infty)$ and for each such extended eigenvalue λ, the corresponding eigenvector can be found in the class of integral operators. It is easy to show that not all such extended eigenvectors A commute with V. Independently, Karaev [4] has obtained the same result in somewhat strengthened form. Unfortunately, this line of attack is not universally available. Namely, Shkarin [11] has shown that there are Volterra operators on a separable Hilbert space with no extended eigenvalues except $\lambda = 1$.

In this article we consider the Volterra integration operator V, $(Vf)(z) = \int_0^z f(t)\,\mathrm{d}t$, on the Wiener algebra

$$W(\mathbb{D}) := \left\{ f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D}) : \|f\| := \sum_{n=0}^{\infty} |\widehat{f}(n)| < \infty \right\}$$

over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where $\widehat{f}(n) = f^{(n)}(0)/n!$ is the nth Taylor coefficient of f. We prove that the set of extended eigenvalues of V is precisely the set $\mathbb{C}\backslash\{0\}$, and describe in terms of Duhamel operators and composition operators the set of corresponding eigenvectors of V.

Recall that for $f, g \in \mathrm{Hol}(\mathbb{D})$ their Duhamel product is defined by

$$(f \circledast g)(z) := \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z f(z-t) g(t)\,\mathrm{d}t = \int_0^z f'(z-t) g(t)\,\mathrm{d}t + f(0) g(z), \tag{4}$$

where the integrals are taken over the segment joining the points 0 and z. It is easy to see that the Duhamel product satisfies all the axioms of multiplication, $\mathrm{Hol}(\mathbb{D})$ is an algebra with respect to $\circledast$ as well, and the function $f(z) \equiv 1$ is the unit element of the algebra $(\mathrm{Hol}(\mathbb{D}), \circledast)$ (see [12]). An operator $\mathscr{D}_f$, $\mathscr{D}_f g := f \circledast g$, is called as the Duhamel operator on $W(\mathbb{D})$.

2. Extended eigenvalues and extended eigenvectors of V

In this section we describe the sets of extended eigenvalues and extended eigenvectors of the Volterra integration operator V on the Wiener algebra $W(\mathbb{D})$.

Note that if $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in W(\mathbb{D})$, then

$$Vf(z) = \int_0^z f(t)\,\mathrm{d}t = \widehat{f}(0) z + \frac{\widehat{f}(1)}{2} z^2 + \frac{\widehat{f}(2)}{3} z^3 + \cdots.$$

From this it is clear that if $Vf(z) = 0$, then

$$\widehat{f}(0) = \widehat{f}(1) = \widehat{f}(2) = \cdots = 0,$$

that is $f = 0$, which shows that $\ker V = \{0\}$. This shows that $\lambda = 0$ is not an extended eigenvalues of the operator V. Therefore, the set of all extended eigenvalues of V lies in $\mathbb{C}\backslash\{0\}$. The following our result shows that the set of extended eigenvalues of V is precisely the set $\mathbb{C}\backslash\{0\}$. We also describe the corresponding extended eigenvectors of V.

Theorem 1. *Let $\lambda \in \mathbb{C}\backslash\{0\}$, and let $A \in \mathscr{B}(W(\mathbb{D}))$ be a nonzero operator.*

(a) *If $|\lambda| \leqslant 1$, then $VA = \lambda AV$ if and only if an operator A has the form $AC_\lambda = \mathscr{D}_{A1}$, where $\mathscr{D}_{A1}$ is the Duhamel operator in $W(\mathbb{D})$ and $(C_\lambda f)(z) = f(\lambda z)$ is a composition operator in $W(\mathbb{D})$.*

(b) *If* $|\lambda|>1$, *then* $VA=\lambda AV$ *if and only if* $A=\mathscr{D}_{A1}C_{1/\lambda}$; *i.e.*,

$$Af(z)=\frac{\mathrm{d}}{\mathrm{d}z}\int_0^z (A1)(z-t)f\left(\frac{t}{\lambda}\right)\mathrm{d}t,\ \ f\in W(\mathbb{D}).$$

Proof.

(a) It is clear from (4) that

$$V^n f=\frac{z^n}{n!}\circledast f,\ \ f\in W(\mathbb{D}), \tag{5}$$

$n=0,1,2,\ldots$. Let $VA=\lambda AV$. Then

$$\lambda^n AV^n=V^nA$$

for each $n\geqslant 0$, that is

$$\lambda^n AV^n f=V^nAf$$

for all $f\in W(\mathbb{D})$, in particular,

$$\lambda^n AV^n1=V^nA1$$

for each $n\geqslant 0$. By considering (5), from this we have

$$A\left(\frac{(\lambda z)^n}{n!}\circledast 1\right)=\left(A1\circledast\frac{z^n}{n!}\right)$$

or

$$\frac{1}{n!}A(\lambda z)^n=\frac{1}{n!}A1\circledast z^n,$$

which shows that

$$Ap(\lambda z)=A1\circledast p(z)$$

for all polynomials $p\in\mathscr{P}$. Since the set $\mathscr{P}$ is dense in $W(\mathbb{D})$ and $(W(\mathbb{D}),\circledast)$ is a Banach algebra (see, for instance, [3,6]), from the last equality we obtain that

$$Af(\lambda z)=A1\circledast f(z)$$

for all $f\in W(\mathbb{D})$. Therefore, $AC_\lambda f=\mathscr{D}_{A1}f$ for all $f\in W(\mathbb{D})$, and hence $AC_\lambda=\mathscr{D}_{A1}$.

Conversely if $AC_\lambda=\mathscr{D}_{A1}$, then we have for each polynomial $p\in\mathscr{P}$ that

$$\begin{aligned}VAp(z)&=VAC_\lambda p\left(\frac{z}{\lambda}\right)=V\mathscr{D}_{A1}p\left(\frac{z}{\lambda}\right)=\mathscr{D}_{A1}Vp\left(\frac{z}{\lambda}\right)=AC_\lambda Vp\left(\frac{z}{\lambda}\right)\\&=AC_\lambda\left(z\circledast p\left(\frac{z}{\lambda}\right)\right)=\lambda AC_\lambda\left(\frac{z}{\lambda}\circledast p\left(\frac{z}{\lambda}\right)\right)=\lambda AC_\lambda(Vp)\left(\frac{z}{\lambda}\right)=\lambda AVp(z)\end{aligned}$$

thus

$$VAp(z)=\lambda AVp(z)$$

for all polynomials p, and hence

$$VAf = \lambda AVf$$

for all $f \in W(\mathbb{D})$. Therefore,

$$VA = \lambda AV,$$

which proves (a).

(b) Suppose that $\lambda AV = VA$. Then,

$$\frac{1}{\lambda}VA = AV$$

and hence

$$\frac{1}{\lambda^n}V^n A = AV^n \tag{6}$$

for all $n \geqslant 0$. By the same arguments, using (6) we can prove that (see the proof of (a))

$$Af(z) = A1 \circledast f\left(\frac{z}{\lambda}\right)$$

for all $f \in W(\mathbb{D})$, which implies that

$$A = \mathscr{D}_{A1}C_{1/\lambda},$$

that is

$$Af(z) = \frac{\mathrm{d}}{\mathrm{d}z}\int_0^z (A1)(z-t)f\left(\frac{t}{\lambda}\right)\mathrm{d}t$$

as desired.

On the other hand, let us now show that an operator A of the form $A = \mathscr{D}_{A1}C_{1/\lambda}$ satisfies the equation

$$\lambda AV = VA.$$

Indeed, for every $f \in W(\mathbb{D})$ we have that

$$\begin{aligned}(AVf)(z) &= (\mathscr{D}_{A1}C_{1/\lambda}Vf)(z) = \mathscr{D}_{A1}(Vf)\left(\frac{z}{\lambda}\right)\\ &= A1 \circledast (Vf)\left(\frac{z}{\lambda}\right) = A1 \circledast \left(\frac{z}{\lambda} \circledast f\left(\frac{z}{\lambda}\right)\right)\\ &= \frac{z}{\lambda} \circledast \left(A1 \circledast f\left(\frac{z}{\lambda}\right)\right) = \frac{z}{\lambda} \circledast \mathscr{D}_{A1}C_{1/\lambda}f(z)\\ &= \frac{1}{\lambda}V\mathscr{D}_{A1}C_{1/\lambda}f(z) = \frac{1}{\lambda}VAf(z),\end{aligned}$$

which completes the proof of (b). Theorem 1 is proved. □

Corollary 1. $\{V\}' = \{\mathscr{D}_f : f \in W(\mathbb{D})\}$, i.e., *the commutant of the Volterra integration operator* $V \in \mathscr{B}(W(\mathbb{D}))$ *is the set of all Duhamel operators on* $W(\mathbb{D})$.

Taking into account that the Duhamel product $\circledast$ is commutative, via Corollary 1 we have that

$$\{V\}'' = \{V\}',$$

where $\{V\}''$ denotes the bicommutant of V.

Recall that a composition operator C_θ, acting in the Wiener algebra $W(\mathbb{D})$ (which is a subalgebra of the disc algebra $C_A(\mathbb{D})$), is defined as

$$(C_\theta f)(z) = (f \circ \theta)(z) = f(\theta(z)),$$

where $\theta : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a suitable analytic function. In the following corollary we are interested in determining whether a composition operator can be an extended eigenvector of V. (Obviously $C_\lambda = C_{\lambda z}$ and $C_{1/\lambda} = C_{z/\lambda}$.)

Corollary 2. *The composition operator* C_θ *is the solution of the equation*

$$VA = \lambda AV,$$

where $|\lambda| \geqslant 1$, *if and only if* $\theta(z) = z/\lambda$.

Proof. Obviously $C_\theta 1 = 1$. Then according to assertion (b) of Theorem 1, we have that

$$VC_\theta = \lambda C_\theta V$$

if and only if

$$C_\theta f(z) = \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z f\left(\frac{t}{\lambda}\right) \mathrm{d}t = f\left(\frac{z}{\lambda}\right) = C_{1/\lambda} f(z)$$

for all $f \in W(\mathbb{D})$, which implies that $C_\theta = C_{1/\lambda}$, that is $\theta(z) = z/\lambda$. The proof of the corollary is completed. $\square$

It turns out that Corollary 2 describes the only situation in which a composition operator C_θ can satisfy

$$VC_\theta = \lambda C_\theta V.$$

In fact, suppose that $VC_\theta = \lambda C_\theta V$ for some λ , $0 < |\lambda| < 1$. Then, according to assertion (a) of Theorem 1, we have that

$$C_\theta C_\lambda = \mathscr{D}_{C_\theta 1} = \mathscr{D}_1 = I,$$

where I is an identity operator in $W(\mathbb{D})$, which implies that $C_\theta C_\lambda z = z$, that is $C_\theta(\lambda z) = z$, or $\lambda\theta(z) = z$, and hence $\theta(z) = z/\lambda$. Therefore,

$$|\theta(1)| = \left|\frac{1}{\lambda}\right| > 1,$$

which contradicts $\theta(1) \in \overline{\mathbb{D}}$.

In conclusion note that the method used above for the Volterra operator acting on the Wiener algebra applies also to other classes of operators, like some weighted shifts on ℓ_p spaces, for instance. Indeed, let us consider the weighted shift operator $Te_n = 1/(n+1)e_{n+1}$, $n \geqslant 0$, on the sequence space ℓ_p $(1 \leqslant p < \infty)$, where $\{e_n\}$ is the standard basis of ℓ_p. For the arbitrarily chosen elements $x = \sum_{n=0}^{\infty} x_n e_n$ and $y = \sum_{n=0}^{\infty} y_n e_n$ of the space ℓ_p, let us define the so-called Duhamel product (see, [12,4]) by the following formula:

$$x \circledast y := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{n!m!}{(n+m)!} x_n y_m e_{n+m}. \tag{7}$$

It is easy to see that formula (7) is correctly defined.

It is also easy to verify that the Duhamel product (7) satisfies all the axioms of multiplication, ℓ_p is the commutative algebra with respect to $\circledast$, an element $e_0 = (1, 0, \ldots)$ is the unit element of the algebra $(\ell_p, \circledast)$, and $Tx = e_1 \circledast x$ for every $x \in \ell_p$, where $e_1 = (0, 1, 0, 0, \ldots)$. An operator $\mathscr{D}_y$, $\mathscr{D}_y x := y \circledast x$, is called the Duhamel operator on ℓ_p. The diagonal operator on ℓ_p with diagonal elements $a_n \in \mathbb{C}$, $n \geqslant 0$, is denoted by D_a, $D_a e_n = a_n e_n$, $n \geqslant 0$.

Now, by the same method, as in the proof of Theorem 1, it can be proved (the proof is omitted) the following theorem which shows that the set of extended eigenvalues of T is the set $\mathbb{C} \backslash \{0\}$.

Theorem 2. *Let $\lambda \in \mathbb{C} \backslash \{0\}$ and let $X \in \mathscr{B}(\ell_p)$ be a nonzero operator. Then we have*

(i) *if $\lambda \in \overline{\mathbb{D}}$, then $TX = \lambda XT$ if and only if an operator X satisfies $XD_\lambda = \mathscr{D}_{Xe_0}$, where D_λ, $D_\lambda e_n = \lambda^n e_n$, $n \geqslant 0$, is the diagonal operator on ℓ_p and $\mathscr{D}_{Xe_0}$ is the Duhamel operator in ℓ_p;*
(ii) *if $\lambda \notin \overline{\mathbb{D}}$, then $TX = \lambda XT$ if and only if $X = \mathscr{D}_{Xe_0} D_{1/\lambda}$.*

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